ON A SPECIAL CASE OF THE TETRAHEDRAL COMPLEX.

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The teacher of line geometry often stands in need of simple examples to illustrate the general theory. The quadratic complex considered in this paper is a special, but interesting, case of the tetrahedral complex, and while the results might be obtained from the general theory the methods here given are worth noting for their simplicity.

For convenience of expression we shall say that a line in space is at right angles to a flat pencil of rays if it is at right angles to that ray of the pencil which it intersects. Using this definition we may state the theorem:

THEOREM. *The system of lines in space at right angles to a given flat pencil forms a quadratic complex of rays; that is, every point in space is the vertex of a quadratic cone the generating lines of which are lines of the system, and every plane in space contains a conic section the tangent lines of which are lines of the system.*

Let A (Fig. 1) be the center of the flat pencil of rays, and let the plane of the pencil be α. Let also a be any ray of A, and let N be the foot of the perpendicular

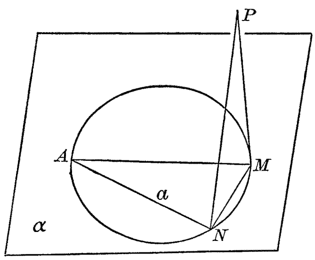

Fig. 1.

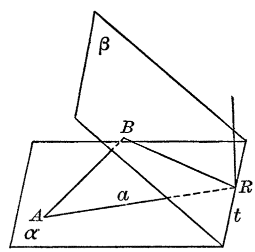

Fig. 2.

from any point P in space upon the line a. Also let M be the foot of the perpendicular from P upon the plane α. Then from elementary geometry ANM is a right angle, and the locus of N is a circle on AM as a diameter. Therefore the rays of the system through P meet the plane α in a circle, and are therefore the generating lines of a quadratic cone.

Further let β (Fig. 2) be any plane in space, and let B be the foot of the perpendicular from A to β. Let t be the line of intersection of α and β, and let R be the point of intersection of any ray a of the pencil A with the line t. Then that line of β which is at right angles to BR and which passes through R is a ray of the system. It follows from the elementary properties of the conic sections that the lines of the system lying in β are tangent to a parabola of which B is the focus, and t the tangent at the vertex. In fact, the lines are seen to be the rays joining corresponding points of two projective point-rows, t and the line at infinity.

Special points of a quadratic complex are such that the cones of rays at these points degenerate into pairs of planes. In the complex described above it is easily seen that the points in α are special. For if P is any point in α then any

line through P in the plane α will be at right angles to some line of A and therefore a line of the system. Also any line through P in a plane at right angles to the line AP will be a line of the system. The cone with vertex P has thus degenerated into a pair of planes. Similarly all the points in the plane at infinity are special. For let P be such a point and a that line of A which is at right angles to PA. Then any line through P in the plane of a and PA belongs to the system. Also any line through P in the plane at infinity belongs to the system. For consider any ray a' of A, and let Q be any point on it. Let a plane at right angles to a' at Q meet PA at P'. Then $P'Q$ is a ray of the system, and if Q moves to infinity P' also moves to infinity in the direction of P. The quadratic cone for any point at infinity is thus seen to degenerate into a pair of planes.

In *Special Planes* of the complex the conic touched by the lines of the system degenerates into a pair of points. This is seen to be the case for any plane through A, and also for any plane through the point at infinity in a direction normal to the plane α. These two points differ from other singular points in that *any* line through them is a line of the system.

Several interesting theorems are easily derived from the above discussion. Thus: *Given two flat pencils in different planes, it is possible to find at most four lines passing through any point of space and at right angles to both pencils.* For the two cones determined by the pencils at any point can have at most four elements in common. An unimportant exception arises when the pencils lie in parallel planes, and the point is chosen on the lines joining their centers. *Given two flat pencils in different planes, it is possible to find at most three finite lines lying in any plane and at right angles to both pencils.* For the parabolas determined by the two complexes in any plane can have at most three tangents in common. Taking account of the infinitely distant elements, however, we may state this and the preceding theorem: *The system of lines at right angles to two flat pencils which lie in different planes is of the fourth order and of the fourth class.*

As a final exercise for the student it is not difficult to prove: *The lines at right angles to two given flat pencils and which meet a given line in space are generators of a ruled quartic surface.*

ON THE USE OF PARTIAL DERIVATIVES IN PLOTTING CURVES FROM THEIR EQUATIONS.

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The article by M. O. Trip in the January, 1914, number of the MONTHLY on "An Application of Partial Derivatives to the Ellipse," has suggested that some further applications which have proved to be practically effective in tracing curves of the second and third degree may be of interest.

If $f(x, y) = 0$ is the equation (rational and integral in x and y) of a conic section, then $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$ are equations of diameters which bisect all chords parallel to the x- and y-axes respectively, and cut the curve at points where